K Pascal's Triangle and K Fibonacci Sequence

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Chapter 1

Pascal's Triangle and Fibonacci Sequence

1.1 Triangle generated from 11^n

Let us consider the power of 11. Then we have the following triangle:

$11^0 =$	1				
$11^1 =$	1	1			
$11^2 =$	1	2	1		
$11^3 =$	1	3	3	1	
$11^4 =$	1	4	6	4	1

Figure 1.1

We can see that the triangle above has a pattern in which each number is equal to the sum of the two numbers right above it. Then, what does we observe in the case $11^5 = 161051$? The digits just overlap as follows:

	1	4	6	4	1
1	5	10	10	5	1

$$
\begin{array}{cccccc}
1 & 5 & & & & \\
 & 1 & 0 & & & \\
 & & 1 & 0 & & \\
 & & & & 5 & 1 \\
\hline
1 & 6 & 1 & 0 & 5 & 1
\end{array}
$$

Figure 1.2

In the case $11^6 = 1771561$ and so forth, we also see the same structure as in the above.

Here, let 10 be represented by x. Hence 11 can be replaced by $x+1$, thus 11^n is $(x+1)^n$. In this way, we obtain the following triangle corresponding to $(x+1)^n$:

$(x+1)^0 \rightarrow$	1							
$(x+1)^1 \rightarrow$	1	1						
$(x+1)^2 \rightarrow$	1	2	1					
$(x+1)^3 \rightarrow$	1	3	3	1				
$(x+1)^4 \rightarrow$	1	4	6	4	1			
$(x+1)^5 \rightarrow$	1	5	10	10	5	1		
$(x+1)^6 \rightarrow$	1	6	15	20	15	6	1	
$(x+1)^7 \rightarrow$	1	7	21	35	35	21	7	1

Figure 1.3

The triangle is called **Pascal's Triangle** which named after Blaise Pascal, a famous Mathematician and Philosopher. Moreover $x + 1$ is called the **generator** of Pascal's Triangle.

Note that the number in the n-th row and k-th column of Pascal's Triangle coincides with the binomial coefficient $\binom{n}{k}$ by the binomial theorem.

$\binom{0}{0}$							
$\binom{1}{0}$	$\binom{1}{1}$						
$\binom{2}{0}$	$\binom{2}{1}$	$\binom{2}{2}$					
$\binom{3}{0}$	$\binom{3}{1}$	$\binom{3}{2}$	$\binom{3}{3}$				
$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$			
$\binom{5}{0}$	$\binom{5}{1}$	$\binom{5}{2}$	$\binom{5}{3}$	$\binom{5}{4}$	$\binom{5}{5}$		
$\binom{6}{0}$	$\binom{6}{1}$	$\binom{6}{2}$	$\binom{6}{3}$	$\binom{6}{4}$	$\binom{6}{5}$	$\binom{6}{6}$	
$\binom{7}{0}$	$\binom{7}{1}$	$\binom{7}{2}$	$\binom{7}{3}$	$\binom{7}{4}$	$\binom{7}{5}$	$\binom{7}{6}$	$\binom{7}{7}$

Figure 1.4

1.2 Pascal's Triangle and Fibonacci Sequence

As already mentioned in Section 1.1, Pascal's Triangle has a triangular pattern of numbers in which each number is equal to the sum of the two numbers right above it. Moreover, Pascal's Triangle tells us an interesting property regarding the sum of a diagonal sequence such as the figure below.

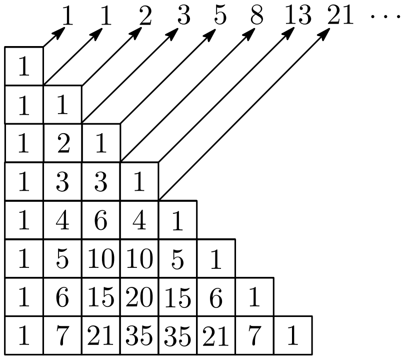


Figure 1.5

The sequence $1, 1, 2, 3, 5, 8, 13, 21, \cdots$ is called the **Fibonacci Sequence** and the numbers in the sequence is called the **Fibonacci Numbers**. The Fibonacci Sequence $\{F_n\}$ satisfies the following difference equation:

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$$

The equation tells us that the every Fibonacci number after the first two is the sum of the last two preceding ones.

1.3 Fibonacci Sequence and Golden Ratio

Let $\{F_n\}$ be the Fibonacci Sequence. Observe that a sequence $\{F_n/F_{n+1}\}$ written by

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \cdots.$$

Then it is well-known that F_n/F_{n+1} converges. In this section, we shall prove the fact for the self-containedness.

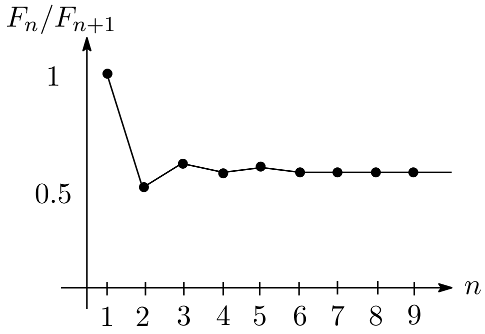


Figure 1.6

Let us first give a precise definition of the limit of a sequence.

Definition of Limit of a Sequence

Let L be a real number. We say the **limit** of a sequence $\{a_n\}$ is L if for any $\varepsilon > 0$, there exists a $M > 0$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > M.$$

Then we call the sequence **converges** to L and write $\lim_{n\to\infty} a_n = L$. Moreover, we say a sequence converges if the limit of the sequence exists, otherwise, the sequence diverges.

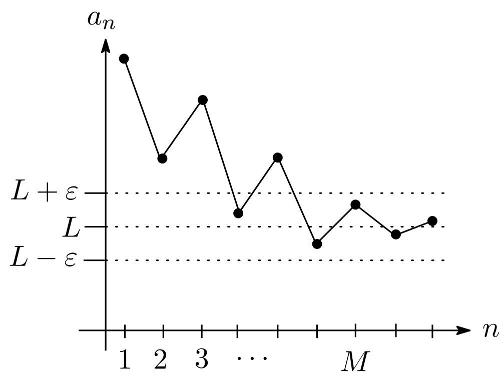


Figure 1.7

When $n > M$, all terms of the sequence lie within ε units of L.

Let us next prove the Squeeze Theorem by using the definition of limit of a sequence.

THEOREM 1. The Squeeze Theorem

Suppose that $\{a_n\}$, $\{b_n\}$ are convergent sequences.
If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then the sequence $\{c_n\}$ converges and

$$\lim_{n\to\infty} c_n = L.$$

Proof Fix $\varepsilon > 0$. Then there exist $M_1 > 0$ and $M_2 > 0$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > M_1$$

and

$$|b_n - L| < \varepsilon \quad \text{whenever} \quad n > M_2.$$

Let M be the largest number of M_1, M_2 and N. Then, we see that $|a_n - L| < \varepsilon$ and $|b_n - L| < \varepsilon$ as long as $n > M$. This implies that

$$-\varepsilon < a_n - L < \varepsilon \quad \text{and} \quad -\varepsilon < b_n - L < \varepsilon,$$

that is,

$$L - \varepsilon < a_n \quad \text{and} \quad b_n < L + \varepsilon.$$

Therefore, combining the above with $a_n \le c_n \le b_n$, one has $L - \varepsilon < c_n < L + \varepsilon$, which yields $|c_n - L| < \varepsilon$. Thus, $\{c_n\}$ converges and $\lim_{n\to\infty} c_n = L$. This completes the proof Q.E.D.

It is important to observe whether a sequence converges. To this end, we shall give some preliminary definitions.

Definition of a Monotonic Sequence

We say that A sequence $\{a_n\}$ is **monotonic**, if its terms are non-decreasing:

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or if its terms are non-increasing:

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$$

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

An important property of the real numbers is that they are **complete**. Roughly speaking, this implies that there are no holes or gaps on the real line. The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, it must have a **least upper bound**. For example, the least upper bound of the sequence $\{a_n\} = \{n/(n+1)\}$ consisting of

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

is 1. The completeness axiom is used in the proof of the following Theorem:

THEOREM 2. Bounded Monotonic Squeeze

If a sequence $\{a_n\}$ is bounded and monotonic, then it coverges.

Proof Assume that the sequence is non-decreasing. For simplicity, we also suppose that each term of the sequence is positive. There exists an upper bound M such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \leq M,$$

since the sequence is bounded. From the completeness axiom, it follows that there is a least upper bound L such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \leq L.$$

For any $\varepsilon > 0$, we have $L - \varepsilon < L$, so that $L - \varepsilon$ is not an upper bound of the sequence. Hence, there exists positive integer N such that $L - \varepsilon < a_N$. By means of the fact the terms of $\{a_n\}$ are non-decreasing, one sees that $a_N \leq a_n$ for all $n > N$. Combining the above, we have $L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$ for every $n > N$, which implies $|a_n - L| < \varepsilon$ for any $n > N$. This yields $\{a_n\}$ converges to L. The case of a non-increasing sequence is similar. The proof is completed. Q.E.D.

PROPOSITION 3.

Let $\{F_n\}$ *be the Fibonacci sequence. Then*

1. $F_{2n-1}F_{2n+1} - (F_{2n})^2 = 1$
2. $(F_{2n-1})^2 - F_{2n-2}F_{2n} = 1$
3. $F_{2n-3}F_{2n} - F_{2n-2}F_{2n-1} = 1$

Proof Let us first prove the Property 1 by the induction on n. For $n = 1$, $F_1F_3 - (F_2)^2 = 1 \times 2 - 1^2 = 1$. This is true. Assume that it holds that

$$F_{2n-1}F_{2n+1} - (F_{2n})^2 = 1$$

for some n. Then we compute

$$\begin{aligned} F_{2(n+1)-1}F_{2(n+1)+1} - (F_{2(n+1)})^2 &= F_{2n+1}F_{2n+3} - (F_{2n+2})^2 \\ &= F_{2n+1}(F_{2n+2} + F_{2n+1}) - (F_{2n+1} + F_{2n})^2 \\ &= F_{2n+1}F_{2n+2} - 2F_{2n+1}F_{2n} - (F_{2n})^2 \end{aligned}$$

$$
\begin{aligned}
&= F_{2n+1}(F_{2n+1} + F_{2n}) - 2F_{2n+1}F_{2n} - (F_{2n})^2 \\
&= (F_{2n+1})^2 - F_{2n+1}F_{2n} - (F_{2n})^2 \\
&= F_{2n+1}(F_{2n} + F_{2n-1}) - F_{2n+1}F_{2n} - (F_{2n})^2 \\
&= F_{2n-1}F_{2n+1} - (F_{2n})^2 \\
&= 1
\end{aligned}
$$

which implies that the Property 1 with $n+1$ holds. Therefore, the Property 1 is true for any n.

Let us show the Property 2. By the Property 1, we see that

$$
\begin{aligned}
1 &= F_{2n-1}F_{2n+1} - (F_{2n})^2 \\
&= F_{2n-1}(F_{2n} + F_{2n-1}) - F_{2n}(F_{2n-1} + F_{2n-2}) \\
&= (F_{2n-1})^2 - F_{2n-2}F_{2n}.
\end{aligned}
$$

Hence the desired property holds. Finally, we shall prove the Property 3. A use of the Property 1 gives us

$$
\begin{aligned}
1 &= F_{2n-3}F_{2n-1} - (F_{2n-2})^2 \\
&= F_{2n-3}(F_{2n} - F_{2n-2}) - F_{2n-2}(F_{2n-1} - F_{2n-3}) \\
&= F_{2n-3}F_{2n} - F_{2n-2}F_{2n-1}.
\end{aligned}
$$

This completes the proof. Q.E.D.

THEOREM 4. Ratio of Consecutive Fibonacci Numbers

Let $\{F_n\}$ be the Fibonacci sequence. Then the sequence $\{F_n/F_{n+1}\}$ converges. Furthermore

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \varphi,$$

where $\varphi = \dfrac{-1+\sqrt{5}}{2}$ *which is called the* **golden ratio**.

Proof Since $\{F_n\}$ is the Fibonacci sequence, we see that

$$F_n \le F_{n+1} = F_n + F_{n-1} \le 2F_n.$$

So, one has

$$\frac{1}{2} \le \frac{F_n}{F_{n+1}} \le 1.$$

From the above, it follows that the sequence $\{F_n/F_{n+1}\}$ is bounded. By the Property 2 in PROPOSITON 3, we compute

$$\begin{aligned} \frac{F_{2n+2}}{F_{2n+3}} - \frac{F_{2n}}{F_{2n+1}} &= \frac{F_{2n+1}F_{2n+2} - F_{2n}F_{2n+3}}{F_{2n+1}F_{2n+3}} \\ &= \frac{F_{2n+1}(F_{2n+1} + F_{2n}) - F_{2n}(F_{2n+2} + F_{2n+1})}{F_{2n+1}F_{2n+3}} \\ &= \frac{(F_{2n+1})^2 - F_{2n}F_{2n+2}}{F_{2n+1}F_{2n+3}} > 0. \end{aligned}$$

This implies that the sequence $\{F_{2n}/F_{2n+1}\}$ is non-decreasing. Similarly, we see that the sequence $\{F_{2n-1}/F_{2n}\}$ is non-increasing. Therefore, THEOREM 2 gives us $\{F_{2n}/F_{2n+1}\}$ and $\{F_{2n-1}/F_{2n}\}$ converge, respectively. Set

$$\lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}} = x, \qquad \lim_{n\to\infty} \frac{F_{2n-1}}{F_{2n}} = y.$$

By the identity $F_{2n+1} = F_{2n} + F_{2n-1}$, we see that

$$\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n}} = \lim_{n\to\infty} \left(1 + \frac{F_{2n-1}}{F_{2n}}\right),$$

which yields

$$\frac{1}{x} = 1 + y.$$

Similarly,

$$\lim_{n\to\infty} \frac{F_{2n}}{F_{2n-1}} = \lim_{n\to\infty} \left(1 + \frac{F_{2n-2}}{F_{2n-1}}\right)$$

implies that

$$\frac{1}{y} = 1 + x.$$

These tell us that

$$x = y, \quad \frac{1}{x} = 1 + x.$$

Hence we conclude that

$$x = \frac{-1 + \sqrt{5}}{2} = \varphi$$

because of $\frac{1}{2} \leq \frac{F_n}{F_{n+1}} \leq 1$. Let us show $\frac{F_n}{F_{n+1}} \to \varphi$ as $n \to \infty$. Denote N by the integer part of $n/2$. If $n = 2k-1$ then $N = k-1$, and the Property 2 of PROPOSITON 3 gives us

$$\frac{F_n}{F_{n+1}} = \frac{F_{2k-1}}{F_{2k}} \geq \frac{F_{2k-2}}{F_{2k-1}} = \frac{F_{2N}}{F_{2N+1}}.$$

Furthermore, it follows from the Property 3 of PROPOSITON 3 that

$$\frac{F_n}{F_{n+1}} = \frac{F_{2k-1}}{F_{2k}} \leq \frac{F_{2k-3}}{F_{2k-2}} = \frac{F_{2N-1}}{F_{2N}}.$$

Therefore, we see that

$$\frac{F_{2N}}{F_{2N+1}} \leq \frac{F_n}{F_{n+1}} \leq \frac{F_{2N-1}}{F_{2N}}.$$

Using the fact

$$\lim_{n\to\infty} \frac{F_{2N}}{F_{2N+1}} = \lim_{n\to\infty} \frac{F_{2N-1}}{F_{2N}} = \varphi,$$

we have

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \varphi.$$

Q.E.D.

Remark that φ is the positive solution of the equation $x + 1 = 1/x$ and $x + 1$ is the generator of Pascal's Triangle. In general, the golden ratio φ is $\frac{1+\sqrt{5}}{2}$. However, we define φ by $\frac{-1+\sqrt{5}}{2}$ in this book.

1.4 Fibonacci Sequence Modulo m

Let us observe how the numbers in the Fibonacci sequence $\{F_n\}$ changes when we reduce them modulo m. Then finite different numbers under modulo m are in the sequence. Indeed, the numbers do not get larger. Here is a list of some examples:

$F_n \pmod 2 : \mathbf{1}, \mathbf{1}, \mathbf{0}, 1, 1, 0, 1, 1, 0, 1, 1, 0, \cdots$
$F_n \pmod 3 : \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{0}, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, \cdots$
$F_n \pmod 4 : \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{0}, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, \cdots$
$F_n \pmod 5 : \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{0}, \mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{2}, \mathbf{4}, \mathbf{1}, \mathbf{0}, 1, 1, 2, 3, 0, \cdots$

We would notice that each Fibonacci sequence in the above eventually repeats. Let us see if this is true. In order to confirm the fact, we need the Pigeonhole principle as follows:.

THEOREM 5. The Pigeonhole Principle

If k is a positive integer and $k+1$ objects are placed into k boxes, then at least one box contains two more objects.

Proof Suppose that none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts that we have $k+1$ objects. Q.E.D.

THEOREM 6. The Fibonacci Sequence Modulo m

The Fibonacci sequence $\{F_n\}$ is periodic modulo any positive integer. Namely, there exist positive integers $T > 1$ and N such that $F_{n+T} = F_n \pmod{m}$ for any n.

Proof Let m be a positive integer and let $\{F_n\}$ be the Fibonacci sequence modulo m. Note that

$$F_n = 0, 1, 2, \cdots, m-1.$$

Let us consider m^2+1 pairs

$$(F_0, F_1),\ (F_1, F_2),\ (F_2, F_3),\ \cdots,\ (F_{m^2}, F_{m^2+1}).$$

Then from the Pigeonhole Principle, there exist integers i and j such that

$$(F_i, F_{i+1}) = (F_j, F_{j+1}) \quad (j \geq i).$$

Because of $F_n = F_{n-1} + F_{n-2}$, $F_i = F_j$ and $F_{i+1} = F_{j+1}$, we obtain

$$F_{i+2} = F_{j+2}.$$

Moreover, combining $F_n = F_{n-1} + F_{n-2}$ with $F_{i+1} = F_{j+1}$ and $F_{i+2} = F_{j+2}$, one sees that

$$F_{i+3} = F_{j+3}.$$

By repeating the above argument, it holds that

$$F_{i+s} = F_{j+s}$$

for any positive integer s. Let $T = j - i$ and $n = i + s$. Then $n + T = j + s$. Therefore,

$$F_n = F_{n+T}$$

for any $n \geq i$. Since it can be allowed to take any $m^2 + 1$ pairs, this implies that $\{F_n\}$ is periodic. Q.E.D.

Remark that the smallest above integer T of the Fibonacci sequence modulo m is called the **Pisano period**. We denote it by $\pi(m)$. The list of $\pi(m)$ for $2 \leq m \leq 12$ is described in the following table.

m	2	3	4	5	6	7	8	9	10	11	12
$\pi(m)$	3	8	6	20	24	16	12	24	60	10	24

It is known that $\pi(m)$ is even if $m > 2$. Moreover, it is also known that $\pi(m) = m$ if and only if $m = 24 \cdot 5^{k-1}$ for some integer $k > 1$.

1.5 Patterns in Pascal's Triangle Modulo p

Let p be a prime number. Let us consider Pascal's Triangle modulo p as follows:

```
        1
       1 1
      1 0 1
     1 1 1 1
    1 0 0 0 1
   1 1 0 0 1 1
  1 0 1 0 1 0 1
 1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 1
```

Pascal's Triangle (mod 2)

```
        1
       1 1
      1 2 1
     1 0 0 1
    1 1 0 1 1
   1 2 1 1 2 1
  1 0 0 2 0 0 1
 1 1 0 2 2 0 1 1
1 2 1 2 1 2 1 2 1
```

Pascal's Triangle (mod 3)

Figure 1.8

1
1 1
1 2 1
1 3 3 1
1 4 1 4 1
1 0 0 0 0 1
1 1 0 0 0 1 1
1 2 1 0 0 1 2 1
1 3 3 1 0 1 3 3 1
1 4 1 4 1 1 4 1 4 1
1 0 0 0 0 2 0 0 0 0 1
1 1 0 0 0 2 2 0 0 0 1 1
1 1 1 0 0 2 4 2 0 0 1 1 1

Pascal's Triangle (mod 5)

Figure 1.9

Let n be the row number of Pascal's Triangle.

Let us consider Pascal's Triangle (mod 2). Then we can find a triangle

1
1 1

between $n = 0$ line and $n = 1$ line of Pascal's Triangle (mod 2). This small triangle is called the 1**st unitary triangle** U_1. Moreover, one sees two 1st unitary triangles

1   1
1 1 1 1 $\longleftrightarrow$ U_1U_1

between $n = 2$ line and $n = 3$ line of Pascal's Triangle (mod 2). We also define a triangle

λ
λ λ

by λU_1 for any integer λ. Then three triangles U_1,$0U_1$,U_1 between $n = 4$ line and $n = 5$ line of Pascal's Triangle (mod 2) can be found. By using λU_1, we describe Pascal's Triangle (mod 2) as follows:

U_1								
U_1	U_1							
U_1	$0U_1$	U_1						
U_1	U_1	U_1	U_1					
U_1	$0U_1$	$0U_1$	$0U_1$	U_1				
U_1	U_1	$0U_1$	$0U_1$	U_1	U_1			
U_1	$0U_1$	U_1	$0U_1$	U_1	$0U_1$	U_1		
U_1	U_1	U_1	U_1	U_1	U_1	U_1	U_1	
U_1	$0U_1$	$0U_1$	$0U_1$	$0U_1$	$0U_1$	$0U_1$	$0U_1$	U_1

Figure 1.10

Figure 1.10 is called **Pascal's Triangle (mod 2) obtained from λU_1**. Replacing U_1 by 1, Pascal's Triangle (mod 2) obtained from λU_1 also would become Pascal's Triangle (mod 2).

In the case of Pascal's Triangle (mod 3), we denote 1st unitary triangle U_1 by the following triangle:

1		
1	1	
1	2	1

between $n = 0$ line and $n = 2$ line of Pascal's Triangle (mod 3).

Similarly, in the case of Pascal's Triangle (mod 5), we define 1st unitary triangle U_1 by the following:

1				
1	1			
1	2	1		
1	3	3	1	
1	4	1	4	1

between $n = 0$ line and $n = 4$ line of Pascal's Triangle (mod 5).

Generally, in the case of Pascal's Triangle (mod p), we denote 1st unitary triangle U_1 by the small triangle between $n = 0$ line and $n = p - 1$ line of Pascal's Triangle (mod p). Hence, we have Pascal's Triangle (mod p) obtained from λU_1.

Let us here prove that Pascal's Triangle (mod p) obtained from λU_1 is also Pascal's Triangle (mod p).

THEOREM 7. Fermat's Little Theorem

Let p be a prime number. Let a be a positive integer with $a \not\equiv 0 \pmod{p}$. Then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Proof First, let us prove the fomula

$$a^p \equiv a \pmod{p}$$

is true for any positive integer a. This formula is clearly true for $a = 0$, which gets our induction started. Suppose that the formula is true for some positive integer a. Since p divides $\binom{p}{k}$ when $0 < k < p$, we have

$$(a+1)^p \equiv a^p + 1^p \pmod{p}.$$

Hence, by using the hypothesis $a^p \equiv a \pmod{p}$, it holds that

$$(a+1)^p \equiv a+1 \pmod{p}.$$

Therefore, the formula $a^p \equiv a \pmod{p}$ is true for all positive integer a. This tells us that p divides $a^p - a = a(a^{p-1} - 1)$. Since p does not divide a by the assumption, p divides $(p^{p-1} - 1)$, so that we conclude that

$$a^{p-1} \equiv 1 \pmod{p}.$$

Q.E.D.

Definition of $O(m)$ extension of a Sequence

For any sequence a_1, a_2, $a_3 \cdots$, a_{n-1}, a_n, a sequence

$$a_1, 0 \cdots 0, a_2, 0 \cdots 0, a_3, 0 \cdots 0, \cdots, 0 \cdots 0, a_{n-1}, 0 \cdots 0, a_n$$

in which m 0's are inserted between a_j and a_{j+1} $(1 \leq j \leq n-1)$ is called the $O(m)$ **extension** of the sequence a_1, a_2, $a_3 \cdots$, a_{n-1}, a_n.

THEOREM 8. Self-Similarity of Pascal's Triangle (mod p)

Let $\{a_k\}_n$ be a sequence on nth row of Pascal's Triangle (mod p). Then $\{a_k\}_{np^t}$ is the $O(p^t - 1)$ extension of a sequence $\{a_k\}_n$, where $t = 1, 2, \cdots$.

Proof Put

$$(x+1)^n = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + 1.$$

Since p divides $\binom{p}{k}$ when $0 < k < p$,

$$\begin{aligned}(x+1)^{np} &= (x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + 1)^p \\ &= x^n + a_1^p x^{(n-1)p} + a_2^p x^{(n-2)p} + \cdots + a_{n-1}^p x^p + 1 \pmod p.\end{aligned}$$

Hence, Fermat's Little Theorem gives us

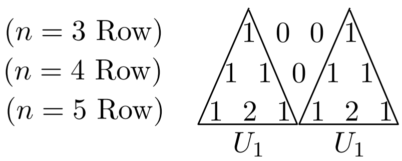
$$(x+1)^{np} = x^n + a_1 x^{(n-1)p} + a_2 x^{(n-2)^p} + \cdots + a_{n-1} x^p + 1 \pmod p.$$

This tells us that $(p-1)$ 0's are inserted between a_j and a_{j+1}. Hence, $\{a_k\}_{np}$ is the $O(p-1)$ extension of a sequence $\{a_k\}_n$. Similarly to the above, in the case $t \geq 2$, it can be proven that $\{a_k\}_{np^t}$ is the $O(p^t - 1)$ extension of a sequence $\{a_k\}_n$. Q.E.D.

Let us observe the self-similarity of Pascal's Triangle (mod 3). For example, as for the sequence $1, 1$ on 1st row of Pascal's Triangle (mod 3), the sequence

$$1, 0, 0, 1$$

on the 3rd row is $O(2)$ extension of a sequence $1, 1$. By the property of Pascal's triangle, we have the following:

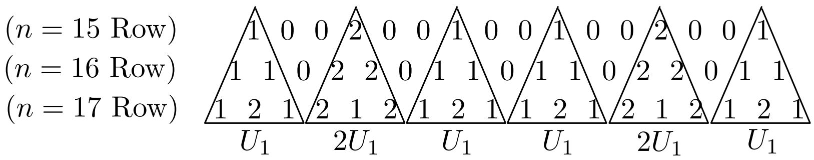


This tells us that U_1, U_1 corresponds to the sequence $1, 1$ on 1st row.

As arguing in the above, the sequence

$$1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1$$

on 15th row is $O(2)$ extension of a sequence $1, 2, 1, 1, 2, 1$ on 5th row, and it can be established that



we also observe that $U_1, 2U_1, U_1, U_1, 2U_1, U_1$ corresponds to the sequence $1, 2, 1, 1, 2, 1$ on 5th row.

In more general, for the sequence$\{a_k\}$ on nth row of Pascal's Triangle (mod p), the sequence on (pn)-th row is $O(p-1)$ extension of a sequence $\{a_k\}$. Moreover, we observe that the sequence $\{a_kU_1\}$ corresponds to the sequence $\{a_k\}$ between (pn)-th row and $(pn+(p-1))$-th row. Hence, the Pascal's Triangle (mod p) obtained from λU_1 is also Pascal's Triangle (mod p).

Chapter 2

3 Pascal's Triangle and 3 Fibonacci Sequence

2.1 Triangle generated from 111^n

Let us consider the power of 111. Then we have the following triangle:

$111^0 =$					1				
$111^1 =$				1	1	1			
$111^2 =$			1	2	3	2	1		
$111^3 =$		1	3	6	7	6	3	1	
$111^4 =$	1	5	1	8	0	7	0	4	1

Figure 2.1

We can see that the triangle above has a pattern in which each number is equal to the sum of the three numbers right above it for 111^2 and 111^3. As for the $111^4 = 151807041$, the digits just overlap as follows:

	1	3	6	7	6	3	1	
1	4	10	16	9	16	10	4	1

```
1 4
  1 0
    1 6 9
        1 6
          1 0 4 1
-------------------
1 5 1 8 0 7 0 4 1
```

Figure 2.2

In the case $111^6 = 1870414552161$ and so forth, we also see the same structure as in the above.

Here, let 10 be represented by x. Hence 111 can be replaced by x^2+x+1, thus 111^n is $(x^2+x+1)^n$. In this way, we obtained the following triangle corresponding to $(x^2+x+1)^n$.

$(x^2+x+1)^0 \to$						1					
$(x^2+x+1)^1 \to$					1	1	1				
$(x^2+x+1)^2 \to$				1	2	3	2	1			
$(x^2+x+1)^3 \to$			1	3	6	7	6	3	1		
$(x^2+x+1)^4 \to$		1	4	10	16	19	16	10	4	1	
$(x^2+x+1)^5 \to$	1	5	15	30	45	51	45	30	15	5	1

Figure 2.3

This triangle is called 3 **Pascal's Triangle** which named after Blaise Pascal, a famous Mathematician and Philosopher. Moreover x^2+x+1 is called the **generator** of 3 Pascal's Triangle.

2.2 3 Pascal's Triangle and 3 Fibonacci Sequence

The 3 Pascal's Triangle has an interesting property in terms of the sum to its diagonal sequence such as the figure below:

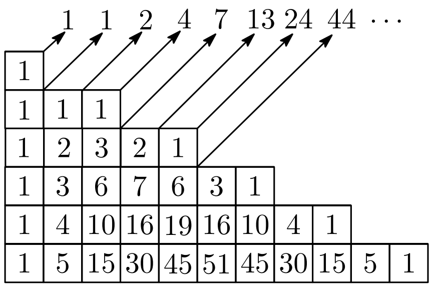


Figure 2.4

The sequence $1, 1, 2, 4, 7, 13, 24, 44, \cdots$ is called the 3 **Fibonacci Sequence**. The Fibonacci Sequence $\{F_n\}$ satisfies the following equation:

$$F_n = F_{n-1} + F_{n-2} + F_{n-3}, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 1.$$

The equation implies that every Fibonacci number after the first three is the sum of the last three preceding ones.

2.3 3 Fibonacci Sequence and Hyper Golden Ratio of dgree 3

Let $\{F_n\}$ be the 3 Fibonacci Sequence. Observe that a sequence $\{F_n/F_{n+1}\}$ written by

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{4}, \frac{4}{7}, \frac{7}{13}, \frac{13}{24}, \frac{24}{44}, \cdots .$$

Does this series $\{F_n/F_{n+1}\}$ converge?

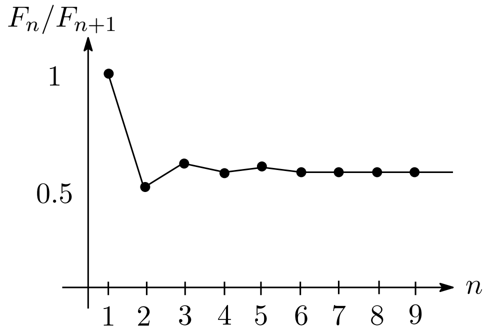


Figure 2.5

Let us first recall the triangle inequality.

THEOREM 9. The Triangle Inequality

For all real numbers x and y, it holds that

$$|x+y| \leq |x| + |y|.$$

Let us here give a definition of the Cauchy sequence.

Cauchy Sequence

We say that a sequence $\{a_n\}$ of real numbers is a **Cauchy sequence**, provided that for every $\varepsilon > 0$, there exists a positive integer N so that $|a_n - a_m| < \varepsilon$ as long as $n, m \geq N$.

The following fact is so-called Cauchy criterion for real sequences.

THEOREM 10. Cauchy Criterion for Real Sequences

A sequence $\{a_n\}$ of real numbers converges if and only if $\{a_n\}$ is a Cauchy sequence.

Proof Firstly, suppose that a sequence $\{a_n\}$ has a finite limit L. Let us show that $\{a_n\}$ is a Cauchy sequence. By the assumption, there exists a positive integer N such that if $n > N$, then $|a_n - L| < \frac{\varepsilon}{2}$. Hence, since $|a_n - L| < \dfrac{\varepsilon}{2}$ and $|a_m - L| < \dfrac{\varepsilon}{2}$ whenever n, $m \geq N$, together with the triangle inequality, we obtain

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| \leq \varepsilon.$$

This implies that $\{a_n\}$ is a Cauchy sequence.

Next, assume that $\{a_n\}$ is a Cauchy sequence, in order to show that $\{a_n\}$ converges. By the assumption, for any $j \geq 1$, there exists an integer $N_j > 0$ such that $|a_n - a_m| < 2^{-j}$ holds whenever $n, m \geq N_j$. Let us define the sequence $\{b_j\}$ by

$$b_j = a_{N_j} - 2^{-j}.$$

Then we have

$$a_n - b_j = a_n - a_{N_j} + 2^{-j} > 0$$

for any $n > N_j$. Hence, one sees that $a_n > b_j$ for every $n > N_j$. Thus, as long as $n > N_j$, each b_j can be a lower bound for elements of the Cauchy sequence $\{a_n\}$. On the other hand, by means of $|a_n - a_{N_j}| < 2^{-j}$, the triangle inequality gives us

$$|a_n - b_j| \leq |a_n - a_{N_j}| + 2^{-j} < 2 \cdot 2^{-j} \tag{2.1}$$

for any $n > N_j$. Further, Noting that $N_j \geq N_1$ for all $j \geq 1$, we deduce that

$$b_1 + 1 - b_j = a_{N_1} - 2^{-1} + 1 - a_{N_j} + 2^{-j} > 2^{-j}$$

for any $j \geq 1$ because of $|a_{N_1} - a_{N_j}| < 2^{-1}$. Hence, for any $j \geq 1$, we have $b_j < b_1 + 1$. Thus $\{b_j\}$ has a least upper bound. Denoting the upper bound by L, we see from the definition of the upper bound that for any $j \geq 1$, there exists $M_j \geq j$ such that $L - b_{M_j} < 2^{-j}$. Further, (2.1) allows us to see that there exists an integer $N_{M_j} > 0$ such that $|a_n - b_{M_j}| < 2 \cdot 2^{-M_j}$ if $n > N_{M_j}$. Therefore, if $n > N_{M_j}$, then it holds that

$$|a_n - L| \leq |a_n - b_{M_j}| + |b_{M_j} - L| < 2 \cdot 2^{-M_j} + 2^{-j} \leq 3 \cdot 2^{-j},$$

which implies that $\{a_n\}$ converges to L. This completes the proof. Q.E.D.

PROPOSITION 11. Property I of 3 Fibonacci Sequence

Let $\{F_n\}$ be the 3 Fibonacci Sequence. Then we have

$$|{F_n}^2 - F_{n-1}F_{n+1}| \leq F_{n+1}.$$

Proof Let us prove by the induction on n. For $n = 1$, we have

$$|{F_1}^2 - F_0F_2| = |1^2 - 0 \cdot 1| = 1 = F_2.$$

This is true. For $n = 2$ and 3, we have

$$|{F_2}^2 - F_1F_3| = |1^2 - 1 \cdot 2| = 1 \leq F_3 = 2,$$
$$|{F_3}^2 - F_2F_4| = |2^2 - 1 \cdot 4| = 0 \leq F_4 = 4.$$

These are also true. Assume that we have the desired inequality for each case n, $n-1$ and $n-2$, that is

$$|F_{n-2}^2 - F_{n-3}F_{n-1}| \leq F_{n-1}$$
$$|F_{n-1}^2 - F_{n-2}F_n| \leq F_n$$
$$|{F_n}^2 - F_{n-1}F_{n+1}| \leq F_{n+1}.$$

For $n+1$, by the assumption of the induction, one sees that

$$\begin{aligned}
&|F_{n+1}^2 - F_{n-1}F_{n+1}| \\
&\quad = |F_{n+1}(F_n + F_{n-1} + F_{n-2}) - F_n(F_{n+1} + F_n + F_{n-1})| \\
&\quad = |F_{n-1}F_{n+1} - {F_n}^2 + F_{n+1}F_{n-2} - F_nF_{n-1}|
\end{aligned}$$

$$
\begin{aligned}
&\leq |F_{n-1}F_{n+1} - {F_n}^2| + |F_{n+1}F_{n-2} - F_nF_{n-1}| \\
&= F_{n+1} + |F_{n+1}F_{n-2} - F_nF_{n-1}| \\
&\leq F_{n+1} + |(F_n + F_{n-1} + F_{n-2})F_{n-2} - (F_{n-1} + F_{n-2} + F_{n-3})F_{n-1}| \\
&= F_{n+1} + |F_nF_{n-2} - F_{n-1}^2 + F_{n-2}^2 - F_{n-3}F_{n-1}| \\
&\leq F_{n+1} + |F_nF_{n-2} - F_{n-1}^2| + |F_{n-2}^2 - F_{n-3}F_{n-1}| \\
&\leq F_{n+1} + F_n + F_{n-1} \\
&= F_{n+2}.
\end{aligned}
$$

This yields the desired inequality in which n replaced by $n+1$. In consequence, we have the desired inequality for all n. Q.E.D.

PROPOSITION 12. Property II of 3 Fibonacci Sequence

Let $\{F_n\}$ be the 3 Fibonacci Sequence. For $m > n$, we have

$$|F_mF_{n+1} - F_nF_{m+1}| \leq F_{m+1}.$$

Proof Let $k = m - n$. Let us show by the induction on k. For $k = 1$, by Proposition 11, we have

$$|F_{n+1}^2 + F_nF_{n+2}| \leq F_{n+2}$$

which is the case $k = 1$ For $k = 2$, by Proposition 11, one has

$$
\begin{aligned}
&|F_{n+2}F_{n+1} - F_nF_{n+3}| \\
&\quad = |(F_{n+1} + F_n + F_{n-1})F_{n+1} - F_n(F_{n+2} + F_{n+1} + F_n)| \\
&\quad = |F_{n+1}^2 - F_nF_{n+2} + F_{n-1}F_{n+1} - {F_n}^2| \\
&\quad \leq |F_{n+1}^2 - F_nF_{n+2}| + |F_{n-1}F_{n+1} - {F_n}^2| \\
&\quad \leq F_{n+2} + F_{n+1} \\
&\quad = F_{n+3}.
\end{aligned}
$$

This is the case $k = 2$. Similarly to the above, for $n = 3$, it holds that

$$|F_{n+3}F_{n+1} - F_nF_{n+4}|$$

$$
\begin{aligned}
&= |(F_{n+2} + F_{n+1} + F_n)F_{n+1} - F_n(F_{n+3} + F_{n+2} + F_{n+1})| \\
&= |F_{n+2}F_{n+1} - F_nF_{n+3} + F_{n-1}^2 - F_nF_{n+2}| \\
&\leq |F_{n+2}F_{n+1} - F_nF_{n+3}| + |F_{n-1}^2 - F_nF_{n+2}| \\
&\leq F_{n+3} + F_{n+2} \\
&= F_{n+4},
\end{aligned}
$$

which is the case $k = 3$. Assume that we have the desired inequality for each case k, $k-1$ and $k-2$, that is

$$
\begin{aligned}
|F_{n+k-2}F_{n+1} - F_nF_{n+k-1}| &\leq F_{n+k-1} \\
|F_{n+k-1}F_{n+1} - F_nF_{n+k}| &\leq F_{n+k} \\
|F_{n+k}F_{n+1} - F_nF_{n+k+1}| &\leq F_{n+k+1}.
\end{aligned}
$$

For $k+1$, it follows from the assumption of the induction that

$$
\begin{aligned}
&|F_{n+k+1}F_{n+1} - F_nF_{n+k+2}| \\
&= |(F_{n+k} + F_{n+k-1} + F_{n+k-2})F_{n+1} - F_n(F_{n+k+1} + F_{n+k} + F_{n+k-1})| \\
&\leq |F_{n+k}F_{n+1} - F_nF_{n+k+1}| + |F_{n+k-1}F_{n+1} - F_nF_{n+k}| \\
&\qquad + |F_{n+k-2}F_{n+1} - F_nF_{n+k-}| \\
&\leq F_{n+k+1} + F_{n+k} + F_{n+k-1} \\
&= F_{n+k+2}
\end{aligned}
$$

which is the desired inequality in which k replaced by $k+1$. Therefore, the desired inequality for all n can be established. Q.E.D.

THEOREM 13. Ratio of Consecutive 3 Fibonacci Numbers

Let $\{F_n\}$ be the 3 Fibonacci sequence. Then the sequence $\{F_n/F_{n+1}\}$ converges. Furthermore it holds that

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \varphi_3,$$

where φ_3 is a positive solution of the equation $x^3 + x^2 + x - 1 = 0$, and called the **hyper golden ratio of degree** 3.

Proof Let $Q_n = \{F_n/F_{n+1}\}$. It is clear that for any $\varepsilon > 0$, there exists a positive integer N such that

$$\frac{1}{F_{n+1}} < \varepsilon.$$

By Proposition 12, for all m, $n > N$, we have

$$\begin{aligned}
|Q_m - Q_n| &= \left| \frac{F_m}{F_{m+1}} - \frac{F_n}{F_{n+1}} \right| \\
&= \frac{|F_m F_{n+1} - F_n F_{m+1}|}{F_{m+1} F_{n+1}} \\
&\leq \frac{F_{m+1}}{F_{m+1} F_{n+1}} \\
&= \frac{1}{F_{n+1}} < \varepsilon.
\end{aligned}$$

Hence, the Cauchy criterion for real sequences tells us that there exists real number φ_3 such that the sequence $\{Q_n\}$ converges to φ_3 as $n \to \infty$. Further, we see from $F_n = F_{n-1} + F_{n-2} + F_{n-3}$ that

$$\begin{aligned}
\frac{F_n}{F_{n-1}} &= 1 + \frac{F_{n-2}}{F_{n-1}} + \frac{F_{n-3}}{F_{n-1}} \\
&= 1 + \frac{F_{n-2}}{F_{n-1}} + \frac{F_{n-3}}{F_{n-2}} \frac{F_{n-2}}{F_{n-1}},
\end{aligned}$$

which implies

$$\frac{1}{Q_{n-1}} = 1 + Q_{n-2} + Q_{n-3} Q_{n-2}.$$

Therefore, the finite limit φ_3 of the sequence $\{A_n\}$ satisfies

$$\frac{1}{\varphi_3} = 1 + \varphi_3 + {\varphi_3}^2.$$

Q.E.D.

2.4 3 Fibonacci Sequence Modulo m

Similarly to the Fibonacci sequence, the 3 Fibonacci sequences modulo m is also periodic. Let us show the fact. Let $\{F_n\}$ be the 3 Fibonacci sequence.

$F_n \pmod 2$: $\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, \cdots$

$F_n \pmod 3$: $\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{0}, 1, 1, 2, 1, 1, 1, 0, 2, 0, 2, 1, 0, 0, \cdots$

$F_n \pmod 4$: $\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{0}, \mathbf{0}, 1, 1, 2, 3, 1, 0, 0, 1, 1, 2, 3, 1, 0, 0, \cdots$

THEOREM 14. The 3 Fibonacci Sequence Modulo m

The 3 Fibonacci sequence $\{F_n\}$ is periodic modulo any positive integer. Namely, there exist positive integers $T > 1$ and N such that $F_{n+T} = F_n \pmod m$ for all n.

Proof Note that

$$F_n = 0, 1, 2, \cdots, m-1.$$

Let us consider $m^3 + 1$ pairs

$$(F_0, F_1, F_2),\ (F_1, F_2, F_3),\ (F_2, F_3, F_4),\ \cdots,\ (F_{m^3}, F_{m^3+1}, F_{m^3+2}).$$

Then there exist integers i and j such that

$$(F_i, F_{i+1}, F_{i+2}) = (F_j, F_{j+1}, F_{j+2}), \quad (j \geq i)$$

by the Pigeonhole Principle. Combining $F_n = F_{n-1} + F_{n-2} + F_{n-3}$ with $(F_i, F_{i+1}, F_{i+2}) = (F_j, F_{j+1}, F_{j+2})$, we have

$$F_{i+3} = F_{j+3}.$$

By repeating the above argument, it turns out that

$$F_{i+s} = F_{j+s}$$

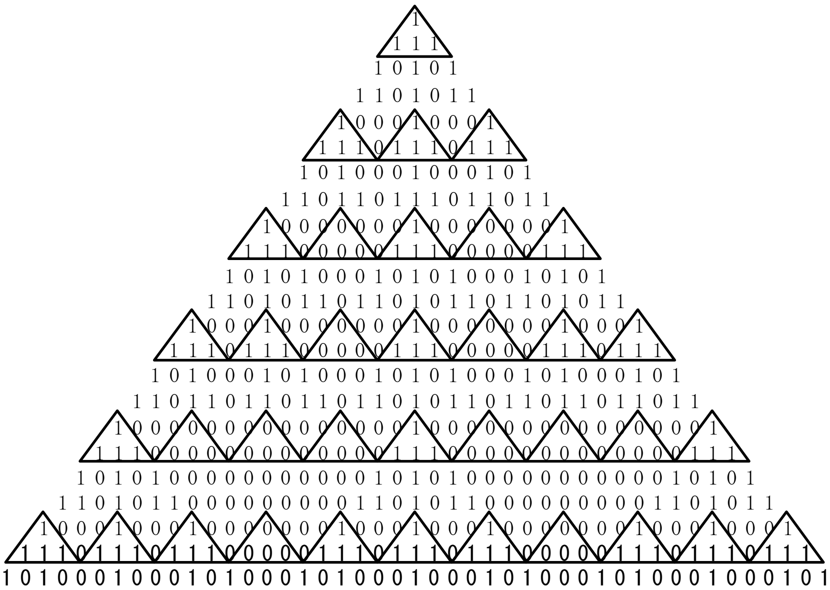
for any positive integer s. Letting $T = j - i$ and $n = i + s$, $n + T = j + s$. Therefore, we have

$$F_n = F_{n+T}$$

for any $n \geq i$. Since it can be allowed to take any $m^3 + 1$ pairs, this implies that $\{F_n\}$ is periodic. Q.E.D.

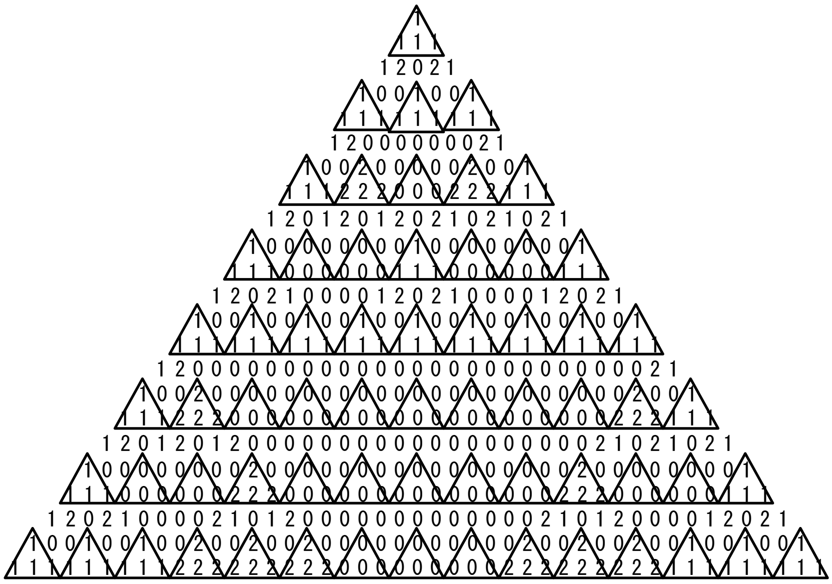
2.5 Patterns in 3 Pascal's Triangle Modulo p

Let p be a prime number. Let us consider 3 Pascal's Triangle modulo p.



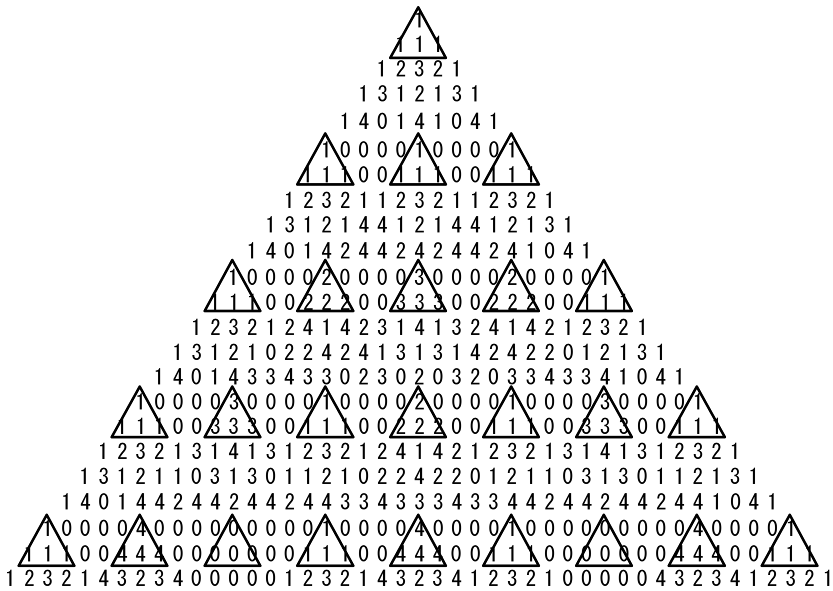
3 Pascal's Triangle (mod 2)

Figure 2.6



3 Pascal's Triangle (mod 3)

Figure 2.7



3 Pascal's Triangle (mod 5)

Figure 2.8

Let n be the row number of 3 Pascal's Triangle.

Similarly to Pascal's Triangle (mod p), We can find a triangle

	1	
1	1	1

between $n = 0$ line and $n = 1$ line of 3 Pascal's Triangle (mod p). This small triangle is called the 1**st unitary triangle** U_1. Moreover, we define a triangle

	λ	
λ	λ	λ

by λU_1 for any positive integer λ.

In the case $p = 2$ and 3, Pascal's Triangle (mod 2) is the following:

n=0 n=1	U_1
n=4 n=5	U_1 U_1 U_1
n=8 n=9	U_1 $0U_1$ U_1 $0U_1$ U_1
n=12 n=13	U_1 U_1 $0U_1$ U_1 $0U_1$ U_1 U_1
n=16 n=17	U_1 $0U_1$ $0U_1$ $0U_1$ U_1 $0U_1$ $0U_1$ $0U_1$ U_1

Figure 2.9

We observe that sequence $\{\lambda U_1\}$ between $4n$ line and $4n+1$ line of 3 Pascal's Triangle (mod 2) in Figure 2.9. This is called 3 **Pascal's Triangle (mod** 2**) obtained from** λU_1. Replacing U_1 with 1, one sees that 3 Pascal's Triangle (mod 2) obtained from λU_1 is also 3 Pascal's Triangle (mod 2).

In the case $p \geq 3$, a sequence $\{\lambda U_1\}$ appears between np line and $np+1$ line of 3 Pascal's Triangle (mod p). Hence, we have 3 Pascal's Triangle (mod p) obtained from λU_1.

We are in position to prove that 3 Pascal's Triangle (mod p) obtained from λU_1 is also 3 Pascal's Triangle (mod p).

THEOREM 15. Self-Similarity of 3 Pascal's Triangle (mod p)

Let $\{a_k\}_n$ be a sequence on nth row of 3 Pascal's Triangle (mod p). Then $\{a_k\}_{np^t}$ is the $O(p^t-1)$ extension of a sequence $\{a_k\}_n$ where $t = 1, 2, \cdots$.

Proof Put

$$(x^2+x+1)^n = x^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_{2n}x + 1.$$

Since p divides $\binom{p}{k}$ when $0 < k < p$,

$$\begin{aligned}(x^2+x+1)^{np} &= (x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + 1)^p \\ &= x^{2n} + a_1^px^{(2n-1)p} + a_2^px^{(2n-2)p} + \cdots + a_{2n-1}^px^p + 1 \\ &\qquad \pmod p\end{aligned}$$

Hence, we see from Fermat's Little Theorem that

$$\begin{aligned}(x^2+x+1)^{np} &= x^{2n} + a_1x^{(2n-1)p} + a_2x^{(2n-2)^p} + \cdots + a_{2n-1}x^p + 1 \\ &\qquad \pmod p,\end{aligned}$$

which implies that $(p-1)$ 0's are inserted between a_j and a_{j+1}. This tells us that $\{a_k\}_{np}$ is the $O(p-1)$ extension of a sequence $\{a_k\}_n$. Similarly, in the case of $t \geq 2$, it can be established that $\{a_k\}_{np^t}$ is the $O(p^t-1)$ extension of a sequence $\{a_k\}_n$. Q.E.D.

Combining Theorem 15 with the same argument as in Pascal's triangle (mod p), we conclude that the 3 Pascal's triangle (mod p) obtained from λU_1 is also the 3 Pascal's triangle (mod p).

Chapter 3

Hyper Golden Ratio of Degree k

3.1 k Pascal's Triangle and k Fibonacci Sequence

Arguing as in the case of the 3 Pascal's triangle, we can define the k **Pascal's triangle** whose generator is $x^{k-1}+x^{k-2}+\cdots x+1$. The k Pascal's triangle has a property in which each number is equal to the sum of the k numbers right above it. Another interesting property of k Pascal's triangle is that the sum of its each diagonal sequences forms into the k **Fibonacci sequence** $\{F_n\}$ defined by

$$F_n = F_{n-1}+F_{n-2}+\cdots+F_{n-k}, \quad F_1 = F_2 = 1, \quad F_k = 0 \ (k \leq 0).$$

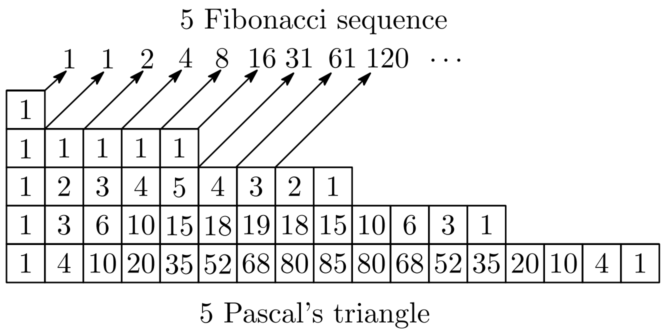


Figure 3.1

Moreover, the sequence $\{F_n/F_{n+1}\}$ converges. Further, we obtain

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \varphi_k \tag{3.1}$$

where φ_k is a positive solution of the equation $x^k + x^{k-1} + \cdots + x - 1 = 0$, called the **hyper golden ratio of degree** k. The identity (3.1) can be proven by using the complex analysis.

3.2 Hyper Golden Ratio of Degree k

If $k \geq 2$ then the equation $x^k + x^{k-1} + \cdots + x - 1 = 0$ has an unique positive solution on interval $(0, 1)$. Let us show the fact. To this end, we need the intermediate value theorem.

THEOREM 16. Intermediate Value Theorem

Let f be a real valued continuous function on the closed interval $[a, b]$ and assume $f(a) < f(b)$. Then for every k such that $f(a) < k < f(b)$, there exists at least one real number c in $[a, b]$ such that $f(c) = k$.

Proof Let us here define the set $H = \{x \in [a, b] : f(x) < k\} \neq \emptyset$. We also denote the supremum of H by c . To show $f(c) = k$, Let us suppose $f(c) < k$. Since f is continuous at c, taking $\varepsilon = k - f(c) > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. This yields $f(c + \delta/2) < k$, so is $c + \delta/2 \in H$. This contradicts the fact c is supremum of H, because of $\delta > 0$. The case $f(c) > k$ can be shown in the same way. Thus we have $f(c) = k$. Q.E.D.

THEOREM 17. Uniqueness Theorem of Hyper Golden Ratio

If $k \geq 2$ then the equation $x^k + x^{k-1} + \cdots + x - 1 = 0$ has an unique positive solution on interval $(0, 1)$.

Proof Let f be the polynomial $x^k + x^{k-1} + \cdots + x - 1$. Note that f is continuous on $(-\infty, \infty)$. If $x_2 \geq x_1 \geq 0$, then it comes from $x_2^j \geq x_1^j$ for all $j \in [1, k]$ that

$$f(x_2) - f(x_1) = ({x_2}^k - {x_1}^k) + ({x_2}^{k-1} - {x_1}^{k-1}) + \cdots + (x_2 - x_1) \geq 0.$$

Thus, f is increasing on $[0, \infty)$. On the other hand, we have

$$f(0) = -1, \quad f(1) > 0.$$

Hence, combining the Intermediate Value Theorem with the monotonicity of f, it is concluded that $x^k + x^{k-1} + \cdots + x - 1 = 0$ has an unique positive solution on interval $(0, 1)$. Q.E.D.

3.3 Coin Tossing and k Fibonacci Sequence

When a fair coin is tossed, there are two possibility:

heads(H) or tails(T)

Assume that the probability of the H is 1/2, and that of T is 1/2.

Question 1

How many times is it expected that we toss the coin until we get two heads in a row?

STEP 1. Assume that two consecutive heads(H) appear for the first time at $n-1$ th and n th times. Then let $H(n)$ be the number of kinds of sequences of heads and tails until we have two heads in a row. For example, $H(1) = 0$ is clear. $H(2) = 1$ from HH, and $H(3) = 1$ from THH. Further, $H(4) = 2$ because of HTHH or TTHH. What is $H(5)$? If the first is T, then one sees TTTHH or THTHH. If H appears at first and T is the second one, then HTTHH is obtained. Thus $H(5) = 2 + 1 = 3$. Similarly, we shall observe $H(6)$. If the first one is T, we have TTTTHH, TTHTHH, or HTTHH. On the other hand, if H appears at first and the second is T, then the sequence is HTHTHH or HTTTHH. Thus $H(6) = 3 + 2 = 5$. We see from the above that the identity

$$H(n) = H(n-1) + H(n-2),$$

which implies that $H(n)$ is the Fibonacci sequence. In what follows, let us denote $H(n)$ by $F(n)$.

The number of kinds of expected sequences from n coin tosses is 2^n. In terms of n coin tosses, the probability $P(n)$ that H appears in a row at $(n-1)$-th and n-th times for the first time is $F(n)/2^n$. Thus the sequence $P(n)$ consists of the following:

$$\frac{0}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{2}{2^4}, \frac{3}{2^5}, \frac{5}{2^6}, \frac{8}{2^7}, \ldots .$$

By the simple argument, we can prove $P(n) = F_{n-1}/2^n$, where F_{n-1} is defined by the difference equation in Section 1.2.

Our purpose is to observe the expected value with respect to the number of the coin tossing until we have two heads in a row

$$E[2] = \sum_{n=1}^{\infty} nP(n).$$

Note that

$$E[2] = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\frac{F_{n-1}}{2^n} = \sum_{n=0}^{\infty}(n+1)\frac{F_n}{2^{n+1}} = \sum_{n=1}^{\infty}(n+1)\frac{F_n}{2^{n+1}}.$$

The following theorem is well-known:

THEOREM 18. D'Alembert Criterion

Assume that the sequence a_n satisfies $a_n > 0$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Then the following holds:

(1) If $r < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $r > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Let $f(x)$ be a power series $\sum a_n x^n$ with $a_n > 0$. We say $R \in [0, \infty]$ is **convergence radius** of $f(x)$ if $f(x)$ converges when $|x| < R$, and $f(x)$ diverges when $|x| > R$.

THEOREM 19. Convergence Radius of a Power Series

Let $\sum a_n x^n$ be a power series with $a_n > 0$, and assume that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} =: R < \infty.$$

Then R is the convergence radius of $\sum_{n=1}^{\infty} a_n x^n$.

Proof Observe that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}|x| = \frac{|x|}{R}.$$

The D'Alembert criterion tells us that then $\sum_{n=1}^{\infty} a_n x^n$ converges if $|x|/R < 1$, and diverges if $|x|/R > 1$. This implies that R is the convergence radius of $\sum_{n=1}^{\infty} a_n x^n$. Q.E.D.

STEP 2. Let us consider a power series $f(x) = \sum_{n=1}^{\infty} F_n x^n$. Then the golden ratio φ is convergence radius of $\sum F_n x^n$ from

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \varphi.$$

Moreover, a direct computation shows

$$\begin{aligned} f(x) &= F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \cdots \\ &= F_1 x + F_1 x^2 + (F_1 + F_2)x^3 + (F_2 + F_3)x^4 + \cdots \\ &= x + x(F_1 x + F_2 x^2 + F_3 x^3 + \cdots) + x^2(F_1 x + F_2 x^2 + F_3 x^3 + \cdots) \\ &= x + (x + x^2) f(x), \end{aligned}$$

which yields

$$\frac{x}{1 - x - x^2} = f(x).$$

The derivative with respect to x of the power series

$$\sum_{n=1}^{\infty} F_n x^{n+1} = xf(x)$$

is

$$\sum_{n=1}^{\infty} (n+1) F_n x^n = f(x) + xf'(x).$$

This gives us

$$\sum_{n=1}^{\infty} (n+1) F_n x^{n+1} = xf(x) + x^2 f'(x).$$

Note that $1/2^n < \varphi$ and $f(x)$ converges as long as $|x| < \varphi$. Hence, we have

$$E[2] = \sum_{n=1}^{\infty} (n+1) F_n \left(\frac{1}{2}\right)^{n+1} = \frac{1}{2} f(1/2) + \frac{1}{2^2} f'(1/2).$$

STEP 3. Let $g(x) = x + x^2$. Then $g'(x) = 1 + 2x$ and

$$f(x) = \frac{x}{1 - g(x)}, \qquad f'(x) = \frac{1 - g(x) + xg'(x)}{(1 - g(x))^2}.$$

Since

$$1 - g(1/2) = \frac{1}{2^2}, \qquad g'(1/2) = 2,$$

we see that

$$f(1/2) = 2, \qquad f'(1/2) = 20.$$

Therefore, one has

$$E[2] = 6 \ (= 2^3 - 2).$$

Question 2

How many times is it expected that we toss the coin until we get three heads in a row?

STEP 1. Assume that three consecutive H first appear between $(n-2)$th and (n)th times. Then let $H(n)$ be the number of kinds of sequences of heads and tails until we have three heads in a row. $H(1) = H(2) = 0$ is clear. $H(n)$ $(3 \leq n \leq 7)$ are the following:

$H(3) = 1$ from HHH.

$H(4) = 1$ from THHH.

$H(5) = 2$ since we have TTHHH and HTHHH.

$H(6) = 2 + 1 + 1 = 4$. Indeed, If T appears at first, then one sees TTTHHH and THTHHH. If H is the first, and the second is T, then HTTHHH is obtained. If H appears at first, H is the second, and T appears at third, we have HHTHHH.

$H(7) = 4+2+1 = 7$. Indeed, If T appears at first, we have TTHTHHH, TTTTHHH, THTTHHH and THHTHHH. If H is the first, and T appears at second, one has HTHTHHH and HTTTHHH. If H appears at first, the second is H, and T appears at third, we obtain HHTTHHH.

In view of the above, one sees the identity

$$H(n) = H(n-1) + H(n-2) + H(n-3),$$

which yields $H(n)$ is 3 Fibonacci sequence. in what follows, set $F(n) = H(n)$. Thus, as for n coin tosses, the probability $P(n)$ that H appears in a row between $(n-2)$-th and n-th times for the first time is $F(n)/2^n$. The sequence $P(n)$ consists of the following:

$$\frac{0}{2}, \frac{0}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{2}{2^5}, \frac{4}{2^6}, \frac{7}{2^7}, \dots .$$

Our purpose is to observe the expected value with respect to the number of the coin tossing until we obtain three heads in a row

$$E[3] = \sum_{n=1}^{\infty} nP(n).$$

Note that

$$E[3] = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\frac{F_{n-2}}{2^n} = \sum_{n=0}^{\infty}(n+2)\frac{F_n}{2^{n+2}} = \sum_{n=1}^{\infty}(n+2)\frac{F_n}{2^{n+2}}.$$

STEP 2. Let us consider a power series $f(x) = \sum F_n x^n$. Then the golden ratio φ_3 of degree 3 is convergence radius of $\sum F_n x^n$ because of

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \varphi_3.$$

Moreover, we calculate

$$\begin{aligned}
f(x) &= F_1x + F_2x^2 + F_3x^3 + F_4x^4 + \cdots \\
&= F_1x + F_2x^2 + (F_1 + F_2)x^3 + (F_1 + F_2 + F_3)x^4 \\
&\quad +(F_1 + F_2 + F_3 + F_4)x^5 \cdots \\
&= x + x(F_1x + F_2x^2 + F_3x^3 + \cdots) + x^2(F_1x + F_2x^2 + F_3x^3 + \cdots) \\
&\quad +x^3(F_1x + F_2x^2 + F_3x^3 + \cdots) \\
&= x + (x + x^2 + x^3)f(x),
\end{aligned}$$

which implies

$$\frac{x}{1 - x - x^2 - x^3} = f(x).$$

The derivative with respect to x of the power series

$$\sum_{n=1}^{\infty} F_n x^{n+2} = x^2 f(x)$$

is

$$\sum_{n=1}^{\infty} (n+2)F_n x^{n+1} = 2xf(x) + x^2 f'(x).$$

Hence, we have

$$\sum_{n=1}^{\infty} (n+2)F_n x^{n+2} = x^2 f(x) + x^3 f'(x).$$

Note that $1/2^n < \varphi_3$ and $f(x)$ is converges if $|x| < \varphi_3$. Therefore, one sees that

$$E[3] = \sum_{n=1}^{\infty} (n+2)F_n \left(\frac{1}{2}\right)^{n+2} = \frac{1}{2^2} f(1/2) + \frac{1}{2^3} f'(1/2).$$

STEP 3. Let $g(x) = x + x^2 + x^3$. Then $g'(x) = 1 + 2x + 3x^2$ and

$$f(x) = \frac{x}{1-g(x)}, \qquad f'(x) = \frac{1-g(x)+xg'(x)}{(1-g(x))^2}.$$

Since

$$1 - g(1/2) = \frac{1}{2^3}, \qquad g'(1/2) = \frac{11}{4},$$

it holds that

$$f(1/2) = 4, \qquad f'(1/2) = 3^3 \times 12.$$

Therefore, we have

$$E[3] = 14 \ (= 2^4 - 2).$$

The same argument gives us the following Theorem:

THEOREM 20. Expected value $E[k]$

The expected value $E[k]$ with respect to the number of the coin tossing until we have k heads in a row is

$$2^{k+1} - 2.$$

Reference

[1] Matsuda,O. and Tsuyama MATH CLUB, 11karahajimarusuugaku, Tokyo Tosyo, 2008, in Japanese.